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Inverse limits with upper semi-continuous set valued bonding functions: an example

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INVERSE LIMITS WITH UPPER SEMI-CONTINUOUS SET VALUED BONDING FUNCTIONS: AN EXAMPLE

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ABSTRACT

While there is a wealth of information pertaining to inverse limits with single valued bonding maps, comparatively little is known about inverse limits with upper semi-continuous set valued bonding functions. In order to add somewhat to the communal knowledge on the subject, this paper provides an example of an inverse limit with a single upper semi-continuous set valued bonding function. It is then shown that the space is a continuum, and its structure is examined via its arc components and through various of its properties, such as dimension and decomposability.

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I wish to thank Dr. Roe for posing an interesting question that, aside from keeping my mind occupied for a substantial period of time, eventually led me to write this thesis. In addition to acquainting me with the underlying mechanics of mathematics, he is also responsible for my transition from a curious student to what one might generously consider a competent mathematician.

My deepest thanks go to Dr. Charatonik for serving as my advisor, for lending me a small amount of his encyclopedic knowledge of inverse limits and continuum theory, and for reading no fewer than ten different drafts of this paper. Without his help, I would not have made it beyond the outline.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

1. INTRODUCTION

Inverse limits as topological spaces were first introduced by Capel in 1954 [2]. In the years following, the efforts of a number of talented mathematicians gave rise to a large suite of tools for dissecting and analyzing such spaces. In the area of continuum theory specifically, there are results pertaining to the dimension, decomposability, and local connectivity of inverse limit spaces, among other properties.

There are also many applications of inverse limits, owing to the concept's close ties to the field of dynamics, but within continuum theory, inverse limits are particularly useful for generating novel examples of continua from relatively simple spaces and functions.

Comparatively little has been discovered about inverse limits with upper semi-continuous set valued bonding functions since they were first defined and described by Ingram and Mahavier in 2006 [8]. For example, it is known that, for inverse limits of continua with single valued bonding maps, the dimension of the inverse limit will not exceed the dimensions of the factor spaces, while Ingram and Mahavier showed it was possible to create an inverse sequence with a single, upper semi-continuous bonding function over the unit interval whose limit is of any finite, non-zero dimension.

Recent results of Nall [12] and Banič [1] represent significant progress in determining the dimension of inverse limits with set valued bonding functions, but, as the example in this paper shows, there is still some work left to be done.

There are also known sufficient conditions under which an inverse limit with single valued bonding maps will be indecomposable, however the space described in this paper shows that different conditions are needed for set valued bonding functions.

Not many properties are known to be preserved by inverse limits with upper semi-continuous set valued bonding functions; Ingram and Mahavier have given conditions under which connectedness and compactness are preserved [8], and Charatonik and Roe have shown, under similar conditions, that the property of trivial shape is preserved [6].

Because there are still very few tools to work with in the area of inverse limits with upper semi-continuous set valued bonding functions, and even fewer of these apply in the specific case of the space described in this paper, many of the properties of the space had to be deduced without such luxuries.

The first properties related in this paper have to do with the connectedness and compactness of the space, its dimension, its embeddability, and its shape. Next, the arc components of the space are described, it is shown that there are uncountably many of them, and that each of them is dense. The last properties to be established are those dealing with the decomposability of the space; specifically, it is shown that the space is not only decomposable, but hereditarily decomposable.

2. DEFINITIONS AND THEOREMS

A *continuum* is a compact, connected metric space. A *map* or *mapping* is a continuous function. A map $f\colon X \to Y$ is said to be *monotone* if, for each connected subset $A \subset Y$, the preimage $f^{-1}(A)$ is connected. If X is a continuum, a map $f\colon X \to Y$ is said to be *confluent* if whenever B is a subcontinuum of Y and F is a component of $f^{-1}(B)$ then $f(F) = B$. Note that monotone maps from one continuum to another are confluent.

The power set 2^X is defined as the set $\{A \subset X \mid A \text{ is closed, non-empty}\}$. A set valued function $f\colon X \to 2^Y$ is called *upper semi-continuous*, if for each closed set $A \subset Y$, the set $\{x \mid f(x) \cap A \neq \emptyset\}$ is closed in X. The *graph* of $f\colon X \to 2^Y$ is the set of points $\{(x, y) \in X \times Y \mid y \in f(x)\}$. [9, §43, I, p. 57]

A *decomposition* $\mathcal{D}$ of a space X is a collection of disjoint subsets of X whose union is the whole space X. The *natural map* $P\colon X \to \mathcal{D}$ maps each point $x \in X$ to the subset $A \in \mathcal{D}$ which contains x. We define the *decomposition topology* τ on $\mathcal{D}$ in such a way as to make the natural map P continuous:

$$\tau = \left\{U \subset \mathcal{D} \mid P^{-1}(U) \text{ is open in } X\right\}.$$

We call the partition $\mathcal{D}$ with the decomposition topology τ the *decomposition space*. Note that the decomposition space $\{\mathcal{D}, \tau\}$ is identical to the quotient space $X/\mathcal{D}$. We say that the decomposition $\mathcal{D}$ is *upper semi-continuous* if the natural map P is upper semi-continuous, and a subset $B \subset X$ is called $\mathcal{D}$-*saturated* if B is a union of elements of $\mathcal{D}$. Note that if $\mathcal{D}$ is upper semi-continuous, the natural map P is closed.

The *arc component* of a point p is the union $\cup\{A \mid A \text{ is an arc, and } p \in A\}$. A continuum X is *decomposable* if there exist two proper subcontinua A and B such that $A \cup B = X$, and a continuum is *hereditarily decomposable* if each of its nondegenerate subcontinua is decomposable. A continuum X is *unicoherent* if for any two proper subcontinua A and B such that $A \cup B = X$, the intersection $A \cap B$ is connected, and a continuum is *hereditarily unicoherent* if all of its subcontinua are unicoherent.

A *dendroid* is an arc connected, hereditarily unicoherent continuum. A λ-*dendroid* is a hereditarily decomposable, hereditarily unicoherent continuum [5, Theorem 1, p. 16]. A *graph* is a continuum which can be represented as a finite union of arcs, any two of which are either disjoint or intersect at one or both endpoints, and a *tree* is an acyclic graph. A continuum is called *tree-like* provided it can be represented as an inverse limit of trees, or equivalently, if there is an ϵ-map for each $\epsilon > 0$ that maps it onto a tree. Note that tree-like continua are hereditarily unicoherent.

In the course of the paper, we will make use of a few theorems. The first is taken from volume two of Kuratowski's Topology [9, §43, I, Theorem 4, p. 58].

Theorem 2.1 (Closed graph theorem)**.** *Let $f\colon X \to 2^Y$. The set valued function f is upper semi-continuous if and only if the graph of f*

$$D = \{(x, y) \in X \times Y \mid y \in f(x)\}$$

is closed.

For the purpose of this paper, *dimension* will refer to the *small inductive dimension.* We use the definition from Engelking's Dimension Theory [7, Ch. 1 §1, Definition 1.1.1, p. 3]:

- $\mathrm{ind}X = -1$ if and only if $X = \emptyset$
- $\mathrm{ind}X \leq n$ where $n = 0, 1, \ldots$, if for every point $x \in X$ and each neighborhood $V \subset X$ of the point x there exists an open set $U \subset X$ such that

  $$x \in U \subset V \quad \textit{and} \quad \mathrm{ind}\partial U \leq n - 1.$$

- $\mathrm{ind}X = n$ if $\mathrm{ind}X \leq n$ and $\mathrm{ind}X > n - 1$.
- $\mathrm{ind}X = \infty$ if $\mathrm{ind}X > n$ for $n = -1, 0, 1, \ldots$

Note that, for the space considered in this paper, the usual definitions of dimension (these being the small inductive, large inductive, and covering dimensions) are equivalent.

The following theorem is also taken from Engelking [7, Ch. 1 §5, Theorem 1.5.3, p. 42].

Theorem 2.2 (The sum theorem)**.** *If a separable metric space X can be represented as the union of a sequence $F_1, F_2, \ldots$ of closed subspaces such that* $\mathrm{ind} F_i \leq n$ *for* $i = 1, 2, \ldots$*, then* $\mathrm{ind} X \leq n$.

We complete our tour of Engelking with one last theorem [7, Ch. 1 §11, Theorem 1.11.4, p. 42].

Theorem 2.3 (The embedding theorem)**.** *Every separable metric space X such that* $0 \leq \dim X \leq n$ *is embeddable in* $\mathbb{R}^{2n+1}$

The next theorem is due to J.J. Charatonik, and can be found in his paper on ramification points [4, 47, p. 239].

Theorem 2.4 (J.J. Charatonik)**.** *No dendroid contains any indecomposable continuum.*

A map $f\colon X \to Y$ is called *essential* if it is not homotopic to any constant map of X into Y. A map is *inessential* if it is not essential. The following theorem, related to inessential maps, is due to Case and Chamberlin [3, Theorem 1, p. 74].

Theorem 2.5. *A given 1-dimensional continuum X is tree-like if and only if every continuous map of X into any graph is inessential.*

The final theorem, due to McLean, can be found in his paper [10].

Theorem 2.6 (B.T. McLean)**.** *The metric confluent image of a tree-like curve is tree-like.*

3. INVERSE LIMITS

While the main focus of this paper is on inverse limits with set valued bonding functions, it will be worth our time to consider briefly those inverse limits with single valued bonding maps, both as a means to familiarize the reader with the concept, and to compare and contrast the known properties in each case.

Suppose that, for each integer $i > 0$ we have a topological space X_i and a map $f_i \colon X_{i+1} \to X_i$. Then we call the pair $\{X_i, f_i\}$ an *inverse sequence*, while the spaces X_i are called *factor spaces*, and the maps f_i are called *bonding maps*. The *inverse limit*, which we denote by $\varprojlim \{X_i, f_i\}$, is defined as follows:

$$\varprojlim \{X_i, f_i\} = \left\{ (x_i) \in \prod_i X_i \mid x_i = f_i(x_{i+1}) \right\}$$

In order to properly orient ourselves, let us consider a few simple examples. In each case, the factor spaces will simply be the unit interval $I = [0,1]$, therefore each inverse limit can be considered as a subset of the Hilbert cube with the distance between any two points given by

$$d(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}.$$

Example 3.1. Let each factor space X_i be the unit interval I and each bonding map be defined thus:

$$f(x) = \begin{cases} 2x & \text{where } 0 \le x \le \frac{1}{2} \\ 1 & \text{where } \frac{1}{2} < x \le 1 \end{cases}.$$

In the inverse limit space, we have a sequence of arcs. The first arc, A_0, is given by

$$A_0 = (a, a/2, a/4, a/8, \ldots) \quad \text{where } a \in [0,1].$$

The nth arc in the sequence will have 1 for the first n coordinates.

$$A_n = (\ldots, 1, c, c/2, \ldots) \quad \text{where } c \in [1/2, 1]$$

Aside from these arcs, the inverse limit also contains a point, $p = (1, 1, 1, \ldots)$.

One endpoint of the arc A_0 is the point $(0, 0, 0, \ldots)$, and the other is $(1, 1/2, 1/4, \ldots)$, which is also an endpoint of A_1. The other endpoint of A_1 is the point $(1, 1, 1/2, 1/4, \ldots)$, which is also an endpoint of A_2. In general, each arc in the sequence is joined to the previous arc at one endpoint and the next arc at the other, and the endpoints of the arcs are approaching the point p. It is clear that the inverse limit in this case is homeomorphic to the unit interval.

The next example is particularly pertinent to the main content of this paper, as we shall see shortly.

Example 3.2. Let each factor space X_i be the unit interval I and each bonding map be defined thus:

$$f(x) = \begin{cases} 2x & \text{where } 0 \leq x \leq \frac{1}{2} \\ 3/2 - x & \text{where } \frac{1}{2} < x \leq 1 \end{cases}.$$

Again, in the inverse limit, we have a sequence of arcs, this time defined in the following manner:

$$S_1 = \left\{ \left(a, \frac{a}{2}, \frac{a}{4}, \frac{a}{8}, \ldots \right) \mid a \in [0, 1] \right\}$$

$$S_2 = \left\{ \left(b, \frac{3}{2} - b, \frac{3}{4} - \frac{b}{2}, \frac{3}{8} - \frac{b}{4}, \ldots \right) \mid b \in \left[\frac{1}{2}, 1 \right] \right\}$$

$$S_n = \left\{ \left(c, \frac{3}{2} - c, c, \frac{3}{2} - c, \ldots, \frac{3}{4} - \frac{c}{2}, \frac{3}{8} - \frac{c}{4}, \ldots \right) \mid c \in \left[\frac{1}{2}, 1 \right] \right\} \quad \text{for } n \text{ odd}$$

$$S_n = \left\{ \left(c, \frac{3}{2} - c, c, \frac{3}{2} - c, \ldots, \frac{c}{2}, \frac{c}{4}, \ldots \right) \mid c \in \left[\frac{1}{2}, 1 \right] \right\} \quad \text{for } n \text{ even}$$

In addition, there is another arc, given by

$$L = \left\{ \left(a, \frac{3}{2} - a, \overline{a, \frac{3}{2} - a}, \ldots \right) \mid a \in \left[\frac{1}{2}, 1 \right] \right\}.$$

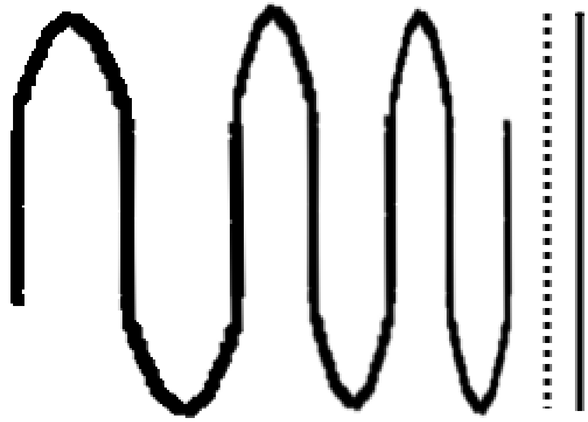

Figure 3.1. The topologist's sine curve.

Again, each arc S_n is joined to S_{n-1} at one endpoint and S_{n+1} at the other, but in this case the arcs S_n approach the arc L. The inverse limit is homeomorphic to the space commonly refered to as the topologist's sine curve, which can be seen in Figure 3.1.

Now that we have a feel for inverse limits with continuous bonding functions, let us turn our attention to the main focus of the paper. For set valued bonding functions, $f_i \colon X_{i+1} \to 2^{X_i}$, the inverse limit is defined as follows:

$$\varprojlim \{X_i, f_i\} = \left\{ (x_i) \in \prod_i X_i \mid x_i \in f_i(x_{i+1}) \right\}$$

In the next example, we examine one of the simplest spaces generated by an inverse limit with a single upper semi-continuous set valued bonding function.

Example 3.3. Let each factor space X_i be the interval I and each bonding map be defined as follows:

$$f(x) = \begin{cases} x & \text{where } 0 \leq x < 1 \\ [0,1] & \text{where } x = 1 \end{cases}$$

In the inverse limit, there is a sequence of arcs defined by

$$A_1 = \{(x, 1, 1, 1, \ldots) \mid x \in [0,1]\},$$

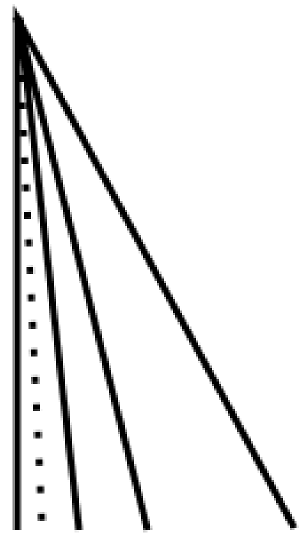

Figure 3.2. The harmonic fan.

$$A_2 = \{(x, x, 1, 1, \ldots) \mid x \in [0, 1]\},$$

$$A_n = \{(x, x, x, x, \ldots, x, 1, 1, \ldots) \mid x \in [0, 1]\}.$$

Additionally, there is another arc,

$$L = \{(x, x, x, x, \ldots) \mid x \in [0, 1]\}.$$

Note that each of the arcs A_n share a common endpoint, $(1, 1, 1, 1, \ldots)$. The other endpoint of A_n has zero for the first n coordinates, and one for all the rest of the coordinates. Also note that the sequence $\{A_n\}$ approaches the arc L.

Using the given metric, we may calculate the distance between the distinct endpoints of any two successive arcs in the sequence. For example, the endpoints which are not shared between A_1 and A_2 are $(0, 1, 1, \ldots)$ and $(0, 0, 1, \ldots)$. The only difference between these occurs at the second coordinate, so the distance between them is $1/4$. In general, the distance between the distinct endpoints of A_{n-1} and A_n is $\frac{1}{2^n}$. The inverse limit in this example is homeomorphic to the so-called harmonic fan, which can be seen in Figure 3.2.

It is well known that, for an inverse sequence of continua with single valued bonding maps, the dimension of the resulting inverse limit will not be greater than the supremum of the dimensions of the factor spaces. However, as we will see in the next example, when dealing with upper semi-continuous set valued bonding functions, there is no such guarantee.

Example 3.4. Let each factor space X_i be the interval I and each bonding map be defined as follows:

$$f(x) = \begin{cases} [0,1] & \text{where } x = 0 \\ 0 & \text{where } 0 < x \leq 1 \end{cases}$$

In order to see that the inverse limit is infinite dimensional, just note that the preimage of zero under the given bonding function is the entire unit interval, and that the preimage of any other point is zero. Because of this, the inverse limit will contain points of the form $(0, x_2, 0, x_4, 0, x_6, \ldots)$, where $x_2, x_4, x_6, \ldots \in [0,1]$. The set of all points of that form make up but a small subset of the inverse limit, but this subset is already homeomorphic to the entire Hilbert cube.

While there is a large suite of tools for dealing with inverse limits with single valued bonding maps, comparatively little is known about inverse limits with set valued bonding functions. Fortunately, we are not left completely in the dark, as there are a few key results which we may use to our advantage. The first of these, due to Ingram and Mahavier, can be found in their paper on inverse limits of upper semi-continuous set valued functions [8, Theorem 4.7, p. 124].

Theorem 3.5 (Ingram, Mahavier)**.** *Suppose that for each i, X_i is a continuum, $f_i \colon X_{i+1} \to 2^{X_i}$ is an upper semi-continuous function, and, for each $x \in X_i$, $f_i(x)$ is connected. Then $\varprojlim \{X_i, f_i\}$ is a continuum.*

Theorem 3.5 is analogous to this next, much older theorem, pertaining to inverse limits with single valued bonding functions, which can be found in Nadler's book [11, Theorem 2.4, p. 19]

Theorem 3.6. *Let $\{X_i, f_i\}$ be an inverse sequence. If, for each i, X_i is a continuum and f_i is a continuous function, then the inverse limit $\varprojlim \{X_i, f_i\}$ is a continuum.*

The next theorem may also be found in Nadler's book [11, Theorem 2.7, p. 21]

Theorem 3.7 (Two Pass Theorem)**.** *Let $\{X_i, f_i\}$ be an inverse sequence where each X_i is a continuum. If, for each i, whenever A_{i+1} and B_{i+1} are subcontinua of X_{i+1} such that $A_{i+1} \cup B_{i+1} = X_{i+1}$, $f_i(A_{i+1}) = X_i$ or $f_i(B_{i+1}) = X_i$, then $\varprojlim \{X_i, f_i\}$ is indecomposable.*

The so-called Two Pass Theorem gives conditions under which the inverse limit is indecomposable. Unfortunately, as we will see later, this theorem does not extend to inverse limits with set valued bonding functions, and, as yet, there is no analogue.

There have, however, been some notable investigations into the dimension of inverse limits with set valued bonding functions, the first of which, undertaken by Banič, produced the following theorem [1, Theorem 6.1, p. 161].

Theorem 3.8. *Let $f\colon [0,1] \to [0,1]$ be a surjective map, and let the upper semi-continuous function $\tilde{f}_t\colon [0,1] \to 2^{[0,1]}$ be the function such that the graph of $\tilde{f}_t$ is the union of the graph of f and the segment $\{t\} \times [0,1]$, for $t \in [0,1]$. Then $\varprojlim\{[0,1], \tilde{f}_t\}$ has dimension 1 or ∞ for all $t \in [0,1]$.*

The next theorem, also related to dimension, is due to Nall and may be found in his paper [12, Theorem 5.3, p. 7].

Theorem 3.9. *If $M = \varprojlim\{X_i, f_i\}$ where each X_i is compact with $\dim X_i \leq m$, and each f_i is an upper semi-continuous set valued function such that $\dim f_i(x) = 0$ for each i, and each $x \in X_{i+1}$, then $\dim M \leq m$.*

In the course of the paper, the concept of *trivial shape* will prove important. Note that, for a continuum X, the following conditions are equivalent:

1. X has trivial shape,

2. X can be written as $X = \bigcap X_n$ where X_n's are contractible continua,

3. X can be written as an inverse limit of contractible continua,

4. For all $\epsilon > 0$ there exists a contractible continuum Y_ϵ and an ϵ map f_ϵ from X onto Y_ϵ.

Furthermore, for 1-dimensional continua, the property of trivial shape is equivalent to that of tree-likeness.

The next theorem, related to trivial shape, is due to Charatonik and Roe [6, Theorem 2].

Theorem 3.10 (W.J. Charatonik, R.P. Roe)**.** *Let* $X_1, X_2, \ldots$ *be a sequence of finite dimensional continua with trivial shape, and let* $f_n \colon X_{n+1} \to 2^{X_n}$ *be upper semi-continuous functions such that* $f_n(x_{n+1})$ *is a continuum with trivial shape. Then* $\varprojlim(X_i, f_i)$ *has trivial shape.*

4. BASIC PROPERTIES AND DIMENSIONALITY

In this paper, we are concerned with the inverse limit space generated by the following set valued function (see Figure 4.1):

$$f(x) = \begin{cases} 2x & \text{where } 0 \leq x \leq \frac{1}{2} \\ 3/2 - x & \text{where } \frac{1}{2} < x < 1 \\ \left[0, \frac{1}{2}\right] & \text{where } x = 1 \end{cases}$$

That is, we consider the space $X = \varprojlim \{I, f\}$.

In this section specifically, we will describe the basic structure of X, along with a few of its distinguishing properties. In particular, we will show that X is a 1-dimensional continuum, and that it is tree-like.

Proposition 4.1. *X is a continuum.*

Proof. First, note that the graph of f is closed, and so by Theorem 2.1, f is upper semi-continuous. Also note that the interval I is a continuum, and that $f(x)$ is either

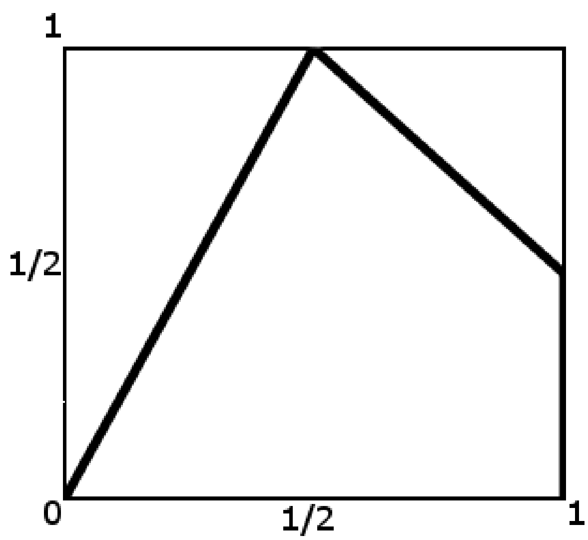

Figure 4.1. Graph of $f(x)$

a point or an arc, both of which are connected sets. Therefore, by Theorem 3.5, X is a continuum. □

Now that we know X is a continuum, we turn our attentnion to its dimension. Although there are two theorems pertaining to the dimension of inverse limits with upper semi-continuous set valued bonding functions, the inverse sequence in our chosen example does not satisfy the conditions of either of these.

Note that $f(1) = [0, 1/2]$, which is not 0-dimensional, so we may not apply Theorem 3.9. With regards to Theorem 3.8, it might be possible to consider the dimension of the inverse limit $X' = \varprojlim\{I, g\}$, where g is defined as

$$g(x) = \begin{cases} 2x & \text{where } 0 \leq x \leq \frac{1}{2} \\ 3/2 - x & \text{where } \frac{1}{2} < x < 1 \\ [0, 1] & \text{where } x = 1 \end{cases}$$

According to Theorem 3.8, X' must have a dimension of one or infinity. First, because the graph of g contains the graph of f, X' must contain X. Similarly, because the graph of f contains the graph of the function given in Example 3.2, we know that X must contain the topologist's sine curve. It is clear, then, that if $\dim X' = 1$, then $\dim X = 1$ as well, while if $\dim X' = \infty$, all we can conclude is that $1 \leq \dim X \leq \infty$, which we already know by virtue of the fact that X is a subset of the Hilbert cube.

Even in the best case scenario, it would still be necessary to show that $\dim X' \neq \infty$, and this could well prove to be a non-trivial task. So, we will forego the use of Theorem 3.8, and show directly that X is 1-dimensional. To that end, we wish to apply Theorem 2.2, so we define a sequence of closed, 1-dimensional subsets of X. First, call the arc, $L = \left\{\left(a, \frac{3}{2} - a, \overline{a, \frac{3}{2} - a}, \ldots\right) \mid a \in \left[\frac{1}{2}, 1\right]\right\}$ the *limit arc*. Next, we define a sequence of arcs:

$$S_1 = \left\{\left(a, \frac{a}{2}, \frac{a}{4}, \frac{a}{8}, \ldots\right) \mid a \in [0, 1]\right\}$$

$$S_2 = \left\{\left(b, \frac{3}{2} - b, \frac{3}{4} - \frac{b}{2}, \frac{3}{8} - \frac{b}{4}, \ldots\right) \mid b \in \left[\frac{1}{2}, 1\right]\right\}$$

$$S_n = \left\{\left(c, \frac{3}{2} - c, c, \frac{3}{2} - c, \ldots, \frac{3}{4} - \frac{c}{2}, \frac{3}{8} - \frac{c}{4}, \ldots\right) \mid c \in \left[\frac{1}{2}, 1\right]\right\} \quad \text{for } n \text{ odd}$$

$$S_n = \left\{ \left(c, \frac{3}{2} - c, c, \frac{3}{2} - c, \ldots, \frac{c}{2}, \frac{c}{4}, \ldots \right) \mid c \in \left[\frac{1}{2}, 1 \right] \right\} \quad \text{for } n \text{ even}$$

We call the ray, $S = \bigcup S_n$ the *leading ray.* Note that $\{S_n\}$ is a sequence of arcs whose limit is L, so $L \cup S$ is a closed, 1-dimensional subset of X.

Now we turn our attention to the more complex subsets. In particular, we let $U_n = \{x \in X \mid x_n = 1\}$ for $n \geq 2$. In order to show that each U_n is closed and 1-dimensional, we first consider sets of these forms:

$$U_{n,p} = \{x \in U_n \mid x_i = p_i, i < n\} \quad \text{for } p \in U_n,$$

$$F_{n,p} = \{x \in U_n \mid x_i = p_i, i \geq n\} \quad \text{for } p \in U_n.$$

Proposition 4.2. *For $n \geq 2$ and $p \in U_n$, $U_{n,p}$ is homeomorphic to the Cantor set.*

Proof. Let $x \in U_{n,p}$. We know, by the definition of U_n that $x_n = 1$. Because $f^{-1}(1) = 1/2$, we also know that $x_{n+1} = 1/2$. From here, there are two possibilities: $f^{-1}(1/2) = \{1, 1/4\}$. So, either $x_{n+2} = 1$ or $x_{n+2} = 1/4$.

In the case where $x_{n+2} = 1$, we know that $x_{n+3} = 1/2$, and then we come back to the same two choices for x_{n+4}; either $x_{n+4} = 1$ or $x_{n+4} = 1/4$.

In the case where $x_{n+2} = 1/4$, there are also two choices. Either $x_{n+3} = 1$ or $x_{n+3} = 1/8$. We can easily visualize all possible outcomes with a recursive tree, seen in Figure 4.2. Each level of the tree represents a coordinate of x, and each path through the tree, starting from the root at the top, enumerates the coordinates x_i, $i \geq n$, for a particular point x in $U_{n,p}$.

From here, we see that $|U_{n,p}| = 2^{\aleph_0}$, and from the metric that $U_{n,p}$ inherits as a subspace of X, it is clear that $U_{n,p}$ is homeomorphic to the Cantor set. $\square$

Proposition 4.3. *For $n \geq 2$ and $p \in U_n$, $F_{n,p}$ is a tree.*

Proof. We begin by examining the structure of $F_{n,p}$ for individual n.

Case 1. $n = 2$

Note that the point $(0, 1, p_3, p_4, \ldots)$ is an element of $F_{2,p}$. It is one endpoint of the arc $(a, 1, p_3, p_4, \ldots)$, where $a \in [0, 1/2]$, with the other endpoint being $(1/2, 1, p_3, p_4, \ldots)$. So $F_{2,p}$ is an arc, as seen in Figure 4.3.

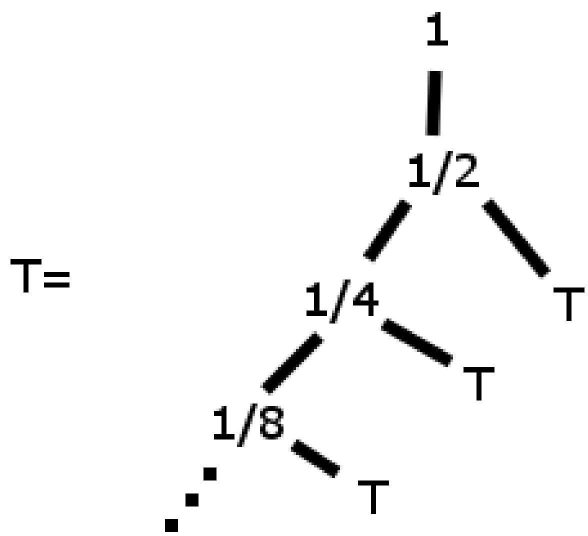

Figure 4.2. A recursive tree describing the coordinates of $x \in U_{n,p}$

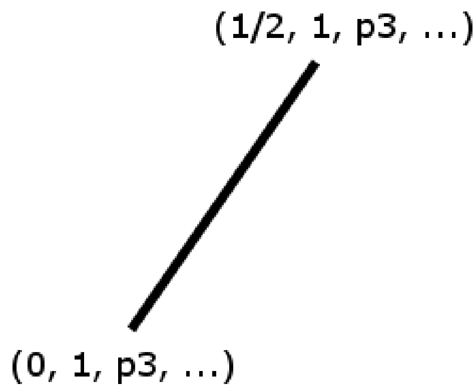

Figure 4.3. $F_{2,p}$

Case 2. $n = 3$

Starting from the point $(0, 0, 1, p_4, \ldots) \in F_{3,p}$, we have the arc $(a, a/2, 1, p_4, \ldots)$, where $a \in [0, 1]$, ending at the point, $(1, 1/2, 1, p_4, \ldots)$. So $F_{3,p}$ is an arc, as seen in Figure 4.4.

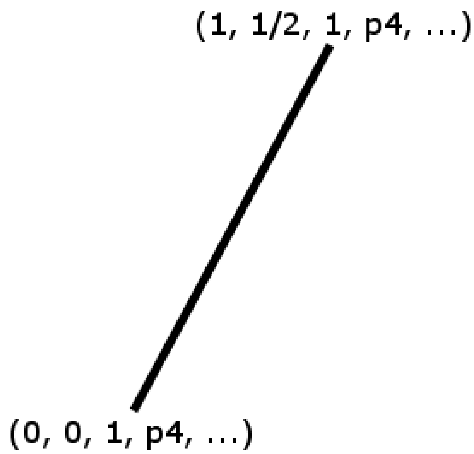

Figure 4.4. $F_{3,p}$

Case 3. $n = 4$

Starting from $(0, 0, 0, 1, p_5, \ldots) \in F_{4,p}$, we have the arc,

$$(a, a/2, a/4, 1, p_5, \ldots)$$

where $a \in [0, 1]$, terminating at $(1, 1/2, 1/4, 1, p_5, \ldots)$. From here we have another arc, $(3/2 - 2b, 2b, b, 1, p_5, \ldots)$, where $b \in [1/4, 1/2]$, terminating at

$$(1/2, 1, 1/2, 1, p_5, \ldots).$$

Finally, from here, there is another arc of the form $F_{2,q}$. So $F_{3,p}$ is an arc once again, as seen in Figure 4.5.

Case 4. $n = 5$

Starting from $(0, 0, 0, 0, 1, p_6, \ldots) \in F_{5,p}$, we have the arc,

$$(a, a/2, a/4, a/8, 1, p_6, \ldots)$$

where $a \in [0, 1]$, terminating at $(1, 1/2, 1/4, 1/8, 1, p_6, \ldots)$.

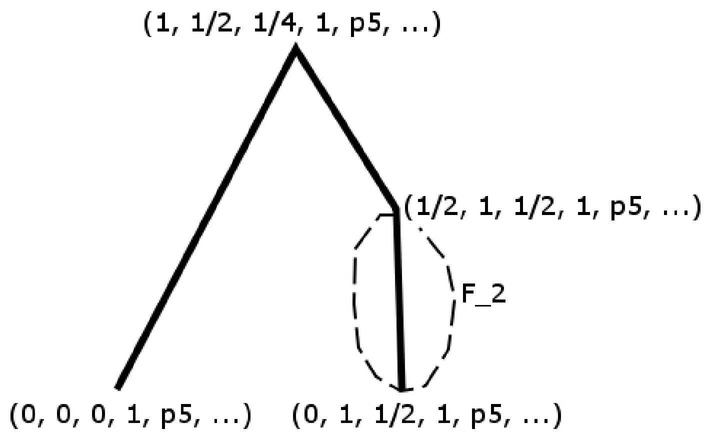

Figure 4.5. $F_{4,p}$

From here we have another arc,

$$(3/2 - 4b, 4b, 2b, b, 1, p_6, \ldots), \text{ where } b \in [1/8, 1/4],$$

terminating at $(1/2, 1, 1/2, 1/4, 1, p_6, \ldots)$.

From this point, there are two arcs: one of the form $F_{2,q}$, and one given by $(2c, 3/2 - 2c, 2c, c, 1, p_6, \ldots)$ where $c \in [1/4, 1/2]$, terminating at

$$(1, 1/2, 1, 1/2, 1, p_6, \ldots).$$

Finally, from here there is another arc of the form $F_{3,r}$. Thus, $F_{5,p}$ is a simple triod, as seen in Figure 4.6.

Case 5. $n = 6$

Starting from $(0, 0, 0, 0, 0, 1, p_7, \ldots) \in F_{6,p}$, we have the arc,

$$(a, a/2, a/4, a/8, a/16, 1, p_7, \ldots)$$

where $a \in [0, 1]$, terminating at

$$(1, 1/2, 1/4, 1/8, 1/16, 1, p_7, \ldots).$$

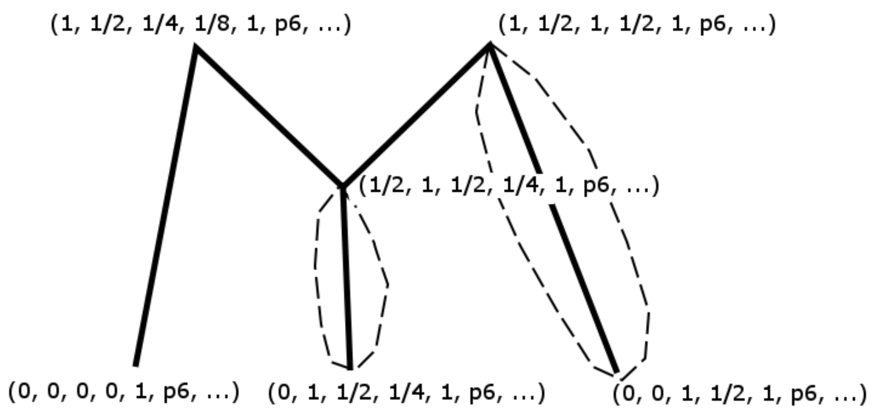

Figure 4.6. $F_{5,p}$

From here we have another arc, $(3/2 - 8b, 8b, 4b, 2b, b, 1, p_7, \ldots)$, where $b \in [1/16, 1/8]$, terminating at $(1/2, 1, 1/2, 1/4, 1/8, 1, p_7, \ldots)$.

From this point, there are two arcs: one of the form $F_{2,q}$, and one given by

$$(4c, 3/2 - 4c, 4c, 2c, c, 1, p_7, \ldots)$$

where $c \in [1/8, 1/4]$, terminating at $(1, 1/2, 1, 1/2, 1/4, 1, p_7, \ldots)$.

From here, again, there are two arcs: one of the form $F_{3,r}$, and one

$$(3/2 - 2d, 2d, 3/2 - 2d, 2d, d, 1, p_7, \ldots)$$

where $d \in [1/4, 1/2]$, terminating at $(1/2, 1, 1/2, 1, 1/2, 1, p_7, \ldots)$.

Finally, from here there is a set of the form $F_{4,s}$. Thus, $F_{6,p}$ is a tree, as seen in Figure 4.7.

Now let us pass to the general case.

Case 6. $n = i$

Here, we note only the significant points; the forms of the connecting arcs have not changed. We start at $(0, 0, 0, 0, 0, 0, \ldots, 1, p_{i+1}, \ldots)$. There is an arc to $(1, 1/2, 1/4, 1/8, 1/16, 1/32, \ldots, 1, p_{i+1}, \ldots)$. From here, there is an arc to

$$(1/2, 1, 1/2, 1/4, 1/8, 1/16, \ldots, 1, p_{i+1}, \ldots).$$

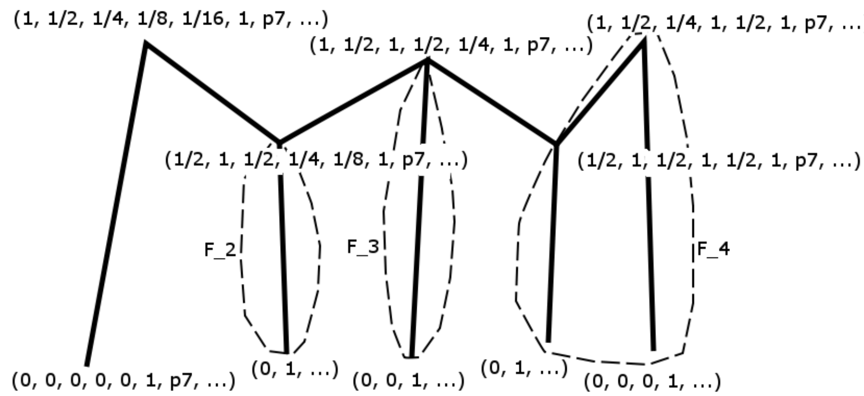

Figure 4.7. $F_{6,p}$

At this point, there two arcs: one of the form F_{2,p^2}, and one to the point

$$(1, 1/2, 1, 1/2, 1/4, 1/8, \ldots, 1, p_{i+1}, \ldots).$$

This pattern continues until we reach the point $(1, 1/2, 1, 1/2, \overline{1, 1/2}, \ldots, 1, p_{i+1}, \ldots)$ in the case where i is odd, or $(1/2, 1, 1/2, 1, \overline{1/2, 1}, \ldots, 1, p_{i+1}, \ldots)$ in the case where i is even. From here, finally, we have a tree of the form $F_{i-2,p^{i-2}}$.

We have shown that $F_{2,p}$ is an arc for $p \in U_2$, and $F_{3,p}$ is an arc for $p \in U_3$. In general, $F_{n,p}$ is a set consisting of a main arc, to which are connected sets of the form F_{i,p^i} for $i = 2, \ldots, n-2$, so we have that $F_{n,p}$ is a tree. $\square$

Now that we have described both $U_{n,p}$ and $F_{n,p}$, we can continue.

Proposition 4.4. *For each n, U_n is 1-dimensional and closed.*

Proof. First, we will show that U_n is homeomorphic to $U_{n,p} \times F_{n,p}$. To see this, consider that points of $U_{n,p}$ have their first n coordinates fixed while the coordinates indexed larger than n vary, and that points of $F_{n,p}$ have their coordinates indexed larger than n fixed while the first n coordinates vary. For a point $x \in U_n$, we can find a point in $U_{n,p}$, namely $(p_1, p_2, \ldots, 1, x_{n+1}, x_{n+2}, \ldots)$, and a point in $F_{n,p}$, $(x_1, x_2, \ldots, 1, p_{n+1}, p_{n+2}, \ldots)$. In a similar way, for any pair of points from $U_{n,p}$ and $F_{n,p}$, we can find exactly one corresponding point in U_n.

Since each $U_{n,p}$ is a Cantor set, and each $F_{n,p}$ is a tree, and because U_n is homeomorpic to $U_{n,p} \times F_{n,p}$, we have that each U_n is a Cantor bundle of trees. U_n is, therefore, 1-dimensional.

In addition, both Cantor sets and trees are compact, so U_n is compact, and thus closed. $\square$

From here it is only a matter of a simple step to get to the result we were looking for.

Proposition 4.5. *X is 1-dimensional.*

Proof. For each point, $x \in X$, there are two possibilities. Either $x_i = 1$ for some $i \geq 2$, in which case, $x \in U_i$, or $p_i \neq 1$ for any $i \geq 2$, in which case, $x \in L$ or $x \in S$.

So $X = [\bigcup_{n=2}^{\infty} U_n] \cup L \cup S$. By Theorem 2.2, X is 1-dimensional. $\square$

Proposition 4.6. *X is non-planar, but may be embedded in three dimensions.*

Proof. To see the former, note that U_5 is a Cantor bundle of simple triods, which is not embeddable in the plane. To see the latter, note that X is a 1-dimensional continuum, so by Theorem 2.3, X is embeddable in $\mathbb{R}^3$. $\square$

Finally, there is one additional property of X worth detailing, and this follows naturally from Proposition 4.5.

Proposition 4.7. *X is tree-like.*

Proof. Since we now know X is 1-dimensional, if we can show that X has trivial shape, then we will have our result. To this end, we will employ Theorem 3.10 and Theorem 2.5.

First, note that the factor spaces, X_i, are just the interval, I, which is obviously finite-dimensional and of trivial shape. Next, note that f is upper semi-continuous, and that, for each $x \in I$, $f(x)$ is either a single point or a closed interval, both of which have trivial shape. By Theorem 3.10, X has trivial shape, and by Theorem 2.5, we have that 1-dimensional continua of trivial shape are tree-like. $\square$

5. ARC COMPONENTS

In this section, we will discuss the arc components of X. Specifically, we will describe the three different types of arc components found in X, show that there are uncountably many arc components, and finally, show that each arc component is dense in X.

We will begin with the following propositions concerning sufficient conditions for two points to be in the same arc component.

Proposition 5.1. *Let $p, q \in X$ and $N > 0$ such that $p_n = q_n$ for all $n \geq N$. Then p and q are in the same arc component.*

Proof. First, let us assume that N is as small as possible while still satisfying the conditions of the proposition.

Note that if $N = 1$, then $p = q$, and the result is obvious. So let us consider the case where $N > 1$.

If $p_N = q_N \in [0, 1/2]$, then $p_{N-1} = q_{N-1} = 2p_N$, so N would not be the smallest possible index satisfying the conditions.

Similarly, if $p_N = q_N \in (1/2, 1)$, then $p_{N-1} = q_{N-1} = 3/2 - p_N$, and again, N would not be the smallest possible index.

So if p_N and q_N are not equal to 1, then N must equal 1, in which case $p = q$ as stated above. If $p_N = q_N = 1$ then, by definition, $F_{N,p} = F_{N,q}$ which is a tree containing both p and q. $\square$

Proposition 5.2. *Let $p, q \in X$ and $N, M > 0$ such that*

$$p_{n+1} = \frac{1}{2} p_n \quad \textit{for} \quad n \geq N$$

and

$$q_{m+1} = \frac{1}{2} q_m \quad \textit{for} \quad m \geq M.$$

Then p and q are in the same arc component.

Proof. Assume that N and M are as small as possible while still satisfying the conditions.

If $p_n \neq 1$ for any n, and $q_m \neq 1$ for any m, then $p, q \in S$, which is a ray, and we are done.

On the other hand, if there is an n such that $p_n = 1$, then we have

$$F_{n,p} \cap S = \{(\ldots, 1/2, 1, 1/2, 1/4, 1/8, \ldots)\}.$$

Then p is in the arc component containing S. So, if either or both of p and q have 1 for some coordinate, in addition to the previously stated properties, then they are in the same arc component. □

Proposition 5.3. *Let $p, q \in X$ and $N, M > 0$ such that*

$$p_{n+1} = 3/2 - p_n \quad \textit{for} \quad n \geq N$$

and

$$q_{m+1} = 3/2 - q_m \quad \textit{for} \quad m \geq M.$$

Then p and q are in the same arc component.

Proof. Assume that N and M are as small as possible while still satisfying the conditions.

If $p_n \neq 1$ for any n, and $q_m \neq 1$ for any m, then $p, q \in L$, which is an arc, and we are done. On the other hand, if there is an n such that $p_n = 1$, then $F_{n,p} \cap L = (\ldots, 1/2, 1, 1/2, 1, 1/2, \ldots)$. Then p is in the arc component containing L.

If either or both of p and q have 1 for some coordinate, then they are in the same arc component. □

Now that we have the conditions for when two points are in the same arc component, we would like to show when two points are in different arc components.

Proposition 5.4. *The arc component containing L and the arc component containing S are disjoint.*

Proof. First, note that, for $x \in X$, if there exists N such that $x_N = 1$, then $x \in F_{N,x}$. If there is no such N, then $x \in S$ or $x \in L$.

Next, note that L and S are themselves disjoint, so if the arc component of L intersects the arc component of S, they must have a common point contained in some $F_{n,x}$. In order to rule out this possibility, let $p \in S$, and $N > 1$ such that $p_N = 1$, and let $q \in L$ such that $q_1 = 1$ or $q_1 = 1/2$.

Now assume that there exists a point x, and $M > 1$ such that $x \in F_{M,p} \cap F_{M,q}$. If $x \in F_{M,p}$, then $x_M = 1$ and $x_{m+1} = 1/2x_m$ for $m \geq M$. On the other hand, if $x \in F_{M,q}$, then $x_M = 1$ and $x_{m+1} = 3/2 - x_m$ for $m \geq M$. It is clear that there is no x that satisfies both of these conditions simultaneously. □

Since it has been established that the arc component containing S and the arc component containing L are disjoint, from here on, we will refer to these arc components as A_S and A_L, respectively.

For the next proposition, we will make use of the set $T(p) = \{n > 1 \mid p_n = 1\}$.

Proposition 5.5. *Let $p \in X \setminus (A_S \cup A_L)$. The arc component of p is $\bigcup_{n \in T(p)} F_{n,p}$.*

Proof. First, note that since p is in neither L nor S, it must be contained in $F_{n,p}$ for some n.

Next, let us choose a point q. We know from Proposition 5.1 that p and q are in the same arc component if there exists N such that $p_n = q_n$ for all $n \geq N$, so let us assume that no such N exists.

If $T(p) \cap T(q) = \emptyset$ then $F_{n,p} \cap F_{m,q} = \emptyset$ for any $n \in T(p)$ and $m \in T(q)$. On the other hand, if there is an $n \in T(p) \cap T(q)$, then $F_{n,p} \cap F_{n,q} = \emptyset$ since for any points $s \in F_{n,p}$ and $t \in F_{n,q}$, there is an m such that $s_m \neq t_m$. □

For the purpose of the next proposition, a point x is an *endpoint* of X if x is an endpoint of every arc containing it. It is clear that the set of endpoints is exactly the set $\{x \in X \mid x_1 = 0\}$.

Proposition 5.6. *The space X contains $2^{\aleph_0}$ different arc components.*

Proof. Consider a point $p \in X \setminus (A_S \cup A_L)$. We know that the arc component of p is $\bigcup_{n \in T(p)} F_{n,p}$.

For each $n \in T(p)$, we know that $F_{n,p}$ contains at most a finite number of endpoints, and therefore that $\bigcup_{n\in T(p)} F_{n,p}$ contains at most a countable number of endpoints.

But if you consider even a small subset of the set of endpoints in X, for example, points of the form $(0, 1, 1/2, \ldots)$, there are uncountably many of them. □

Proposition 5.7. *Each arc component of X is dense in X.*

Proof. Given a point p in one arc component, we can choose a point q from any other arc component such that for any N, $q_n = p_n$ for all $n \leq N$. □

Now that we know something of the properties of the arc components, let us turn our attention to their forms. As was already stated, the vast majority of arc components (that is, all but A_S and A_L) are of the form

$$\bigcup_{n\in T(p)} F_{n,p} \quad \text{for } p \in X \setminus (A_S \cup A_L).$$

It is simple to see, from the previous results, that the arc component containing S is exactly the union

$$A_S = \left(\bigcup \{F_{n,p} \,|\, p \in S, n \geq 2, p_n = 1\}\right) \cup S,$$

and that the arc component containing L is exactly the union

$$A_L = \left(\bigcup \{F_{n,p} \,|\, p \in L, n \geq 2, p_n = 1\}\right) \cup L.$$

The final proposition in this section will be useful in the next section.

Proposition 5.8. *Let A be an arc component of X such that $A \neq A_L$. Then there is a subset $Q \subset A$ such that $L \subset \overline{Q}$.*

Proof. First if $A = A_S$ then we will take $Q = S$, and the result is obvious. So let us assume that $A \neq A_S$, and choose a point $p \in A$ which we will use to define Q.

Recall that $T(p) = \{n_i\}$ is the set of indices such that the coordinates $p_{n_i} = 1$. Let the point q^0 be the endpoint $(0, 0, 0, \ldots, 1, \ldots)$, where $q^0_{n_1} = 1$, all previous coordinates are zero, and all subsequent coordinates are the same as those of p, and let $q^1 = (1, 1/2, 1/4, \ldots, 1, \ldots)$. Denote the arc between q^0 and q^1 by Q_1.

In general, q^m can be described thus:

- The coordinates q^m_n alternate between 1 and $1/2$ for $n < m$. For m odd, $q^m_1 = 1$, and for m even, $q^m_1 = 1/2$, and in both cases $q^m_n = 3/2 - q^m_{n-1}$.
- The coordinate $q^m_m = 1$.
- The coordinates q^m_n are of the form $(\ldots, 1/2, 1/4, 1/8, \ldots)$ for $m < n < n_i$ where n_i is the smallest index in $T(p)$ greater than m.
- The coordinate $q^m_{n_i} = 1$.
- The coordinates $q^m_n = p_n$ for $n_i < n$.

Next, we denote the arc between q^{m-1} and q^m by Q_m, and we define Q to be the union $\bigcup Q_m$. To see that L is in the closure of Q, note that for any point $r \in L$ and any $N > 0$, we can find a point $s \in Q$ such that $s_n = r_n$ for $n < N$. $\square$

In simpler terms, Proposition 5.8 shows that each arc component, aside from A_L, contains a ray that winds down on L.

6. DECOMPOSABILITY

Recall that the Two Pass Theorem (Theorem 3.7) gives conditions under which an inverse limit is indecomposable. We will see that this theorem does not extend to inverse limits with set valued bonding functions, as the continuum X is, in fact, hereditarily decomposable.

In order to show this, we define the decomposition $\mathcal{D} = L \cup \{\{x\} \mid x \in X \setminus L\}$. Note that $\mathcal{D}$ is an upper semi-continuous decomposition of X. To see this, first consider the partition $\mathcal{D}$ as a subset of 2^X, and the natural map to be $P\colon X \to 2^X$. If we take a closed subset $A \in 2^X$, the set $\{x \mid P(x) \cap A \neq \emptyset\}$ is exactly the set $\{x \mid P(x) \subset A\}$ which is simply the preimage $P^{-1}(A)$. Because P is continuous, $P^{-1}(A)$ is a closed set. Also note that the natural map P is monotone, because essentially its only effect is to shrink L to a point.

Proposition 6.1. *The decomposition space $X/\mathcal{D}$ is a dendroid.*

Proof. First, note that since the natural map $P\colon X \to X/\mathcal{D}$ is monotone, it is confluent. By Theorem 2.6, $X/\mathcal{D}$ is tree-like.

Next, from Proposition 5.8, we know that each arc component A of X, aside from A_L, contains a ray Q such that $L \subset \overline{Q}$. Since P is continuous, we have that $P(L) \in P(\overline{Q}) \subset \overline{P(Q)}$. Of course, the image $P(Q)$ is again a ray, but the closure $\overline{P(Q)}$ is an arc containing $P(L)$. Since this is true for each such Q, $X/\mathcal{D}$ is arc connected.

Since $X/\mathcal{D}$ is arc connected and tree-like, and tree-like continua are hereditarily unicoherent, it is necessarily a dendroid. □

Proposition 6.2. *The space X is hereditarily decomposable.*

Proof. Let A be a subcontinuum of X. Note that because the decomposition space $X/\mathcal{D}$ is a dendroid, by Theorem 2.4, it is hereditarily decomposable. Since P is continuous and closed, $P(A)$ is a subcontinuum of $X/\mathcal{D}$, and therefore decomposable.

First, let us consider the case where $A \cap L = \emptyset$ or $L \subset A$. It is clear that A is $\mathcal{D}$-saturated. Take two proper subcontinua, K_1 and K_2, of $P(A)$ such that $K_1 \cup K_2 = P(A)$. By monotonicity and continuity of P, the preimages $P^{-1}(K_1)$ and

$P^{-1}(K_2)$ are both continua, $A = P^{-1}(K_1) \cup P^{-1}(K_2)$, and $P^{-1}(K_1)$, $P^{-1}(K_2)$ are proper subcontinua of A. We conclude that in the case where A is $\mathcal{D}$-saturated, A is decomposable.

Next, let us consider the case where A is not $\mathcal{D}$-saturated, i.e. where $A \cap L \neq \emptyset$ and $L \not\subset A$. We aim to show that A is a particular subset of A_L, and that it is homeomorphic to $P(A)$ unioned with an arc. To this end, assume there is a point p such that $p \in A \setminus A_L$.

First, since $p \notin A_L$, then we know that the arc component of p contains a ray whose closure contains L, and any continuum containing such a ray would necessarily contain the whole set L. Since $L \not\subset A$, A cannot contain any such ray.

The other possibility is that there is a sequence of points $\{p^n\}$ in A_L which converges to p. Since for any two points of the sequence, p^i and p^j, we can find a neighborhood U containing p^i and not p^j, in order to maintain connectivity, A must contain arcs $A_{i,j} \subset A_L$ where p^i, $p^j \in A_{i,j}$. However, close inspection reveals that $L \subset \overline{\bigcup_{i,j} A_{i,j}}$. Once again, since $L \not\subset A$, A cannot contain any such sequence. Note that we arrive at a similar result if we consider a sequence $\{q^n\}$ in the arc component of p which converges to q.

Next, let us define the open interval L' as follows:

$$L' = \left\{ \left(a, \frac{3}{2} - a, \overline{a, \frac{3}{2} - a}, \dots \right) \mid a \in \left(\frac{1}{2}, 1 \right) \right\}$$

Note that the set $A_L \setminus L'$ has two arc components, call them A_{L1} and A_{L2}, and similarly to the above cases, it can be shown that A cannot contain points of both of these arc components without also containing L.

So if A is not $\mathcal{D}$-saturated, A is either a subset of $A_{L1} \cup L$ or $A_{L2} \cup L$, in which case A is homeomorphic to $P(A)$ joined with an arc at $P(L)$, and is therefore decomposable. $\square$

The next proposition follows immediately.

Proposition 6.3. *X is a λ-dendroid.*

Moreover, we can say that the decomposition $\mathcal{D}$ is the finest decomposition such that the decomposition space $X/\mathcal{D}$ is a dendroid.

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